MATH 207(c)–Analysis IV

1. Compute the real and imaginary parts of the following functions. Then, using the Cauchy–Riemann equations, proceed to show that they are holomorphic over their domain of definition and calculate their (complex) derivative.

(a) $f(z) = e^z$ over $\mathbb{C}$.

(b) $f(z) = z^3$ over $\mathbb{C}$.

(a) $f(z) = \frac{1}{z^2}$ over $\mathbb{C}^*$.

2. Show that the following functions are entire and calculate their derivative.

(a) $\cos(z) = \frac{e^{iz}+e^{-iz}}{2}$,

(b) $\cosh(z) = \frac{e^{z}+e^{-z}}{2}$,

(c) $\sinh(z) = \frac{e^{z}-e^{-z}}{2}$.

3. Is the function $f(z) = \big(\mathrm{Re}(z)\big)^2$ holomorphic over $\mathbb{C}$? Justify your answer.

4. Let $f : \Omega \to \mathbb{C}$, $f(x+yi) = u(x,y) + v(x,y)i$, be a holomorphic function, where $\Omega \subset \mathbb{C}$ is an open domain and u, v are the real and imaginary parts of f, respectively. Show that u, v are *harmonic* functions on Ω, i.e. they satisfy

$$\Delta u = 0 \quad \text{and} \quad \Delta v = 0,$$

where $\Delta \doteq \frac{\partial^2}{\partial x^2} + \frac{\partial^2}{\partial y^2}$.

5. Let $f : \mathbb{C} \to \mathbb{C}$, $f(x+yi) = u(x,y) + v(x,y)i$, be a holomorphic function. Let (r,θ) be the *polar* coordinates on $\mathbb{C} \setminus \{0\}$, defined by the relations $x = r\cos(\theta)$ and $y = r\sin(\theta)$. Show that, when expressed in the polar coordinate system, the Cauchy–Riemann equations become

$$\frac{\partial u}{\partial r} = \frac{1}{r}\frac{\partial v}{\partial \theta} \quad \text{and} \frac{\partial v}{\partial r} = -\frac{1}{r}\frac{\partial u}{\partial \theta}.$$

(Hint: Examine how the derivatives $\frac{\partial}{\partial x}$ and $\frac{\partial}{\partial y}$ transform under the cooridnate transformation $(x,y) \to (r,\theta)$.)

6. Let $f : \mathbb{C} \to \mathbb{C}$, $f(x+iy) = u(x,y) + v(x,y)i$, be an entire function, such that its real part is given by

$$u(x,y) = e^{(x^2-y^2)}\cos(2xy).$$

Can you find the expression for $v(x,y)$?